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Mathematical aspects of trapping modes in the theory of surface waves

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A horizontal canal of infinite length and of constant width and depth contains inviscid fluid under gravity. The fluid is bounded internally by a submerged horizontal cylinder which extends right across the canal and has its generators normal to the sidewalls. Suppose that the fluid is set in motion by a surface pressure varying across the canal, then some of the energy is radiated to infinity while some of the energy is trapped in characteristic modes (bound states) near the cylinder. The existence of trapping modes in special cases was shown by Stokes (1846) and Ursell (1951); a general treatment, given by Jones (1953), is based on the theory of elliptic partial differential equations in unbounded domains. In the present paper a much simpler treatment is given which uses only the theory of bounded symmetric linear operators together with Kelvin's minimum-energy theorem of classical hydrodynamics.

1. Introduction

Consider water initially at rest in a rectangular box, and suppose that it is set in motion by a sudden localized impulsive pressure acting on the free surface. According to the linear theory of water waves in a frictionless fluid (see Lamb 1932, chapter 9) the subsequent motion is composed of an infinite discrete sum of characteristic modes each of which continues to oscillate indefinitely with its own characteristic frequency. (We say that the frequency spectrum is discrete.) In a canal of rectangular cross-section and infinite length, on the other hand, the waves carry the energy to infinity and so the wave motion at any point decays to zero. (We say that the frequency spectrum is continuous.) If such a canal contains fixed submerged bodies of finite volume, will the wave motion decay to zero, or will there also be discrete modes (trapping modes or bound states) with characteristic frequencies? The answer to this question is in general not known. In the present paper we shall consider a simpler problem: we shall henceforth suppose that the internal boundary of the fluid is a horizontal cylinder, with generators perpendicular to the sidewalls and extending from wall to wall. Then it is known that trapping modes can exist in certain cases. The first trapping mode, discovered by Stokes (1846), is a mode on a sloping beach; it was shown by Ursell (1951) that a trapping mode exists for a submerged circular cylinder of sufficiently small radius; soon afterwards Jones (1953) gave arguments based on variational principles that a trapping mode exists for a submerged cylinder of any cross-section that is symmetrical about a vertical plane. Jones's arguments are based on the theory of elliptic partial differential equations applied to unbounded domains and thus involve profound mathematical concepts. It is the purpose of the present work to show that the same problem can be treated much more simply by reducing it to the solution of an integral equation with a symmetric kernel. It will be shown that the corresponding integral operator is a bounded symmetric operator with a spectrum that is partly continuous and partly discrete. The existence of a trapping mode for a cylinder of any cross-section then follows from the corresponding result for a circular cross-section of small radius by use of a variational principle which is in fact Kelvin's minimum-energy theorem of classical hydrodynamics.

2. Statement of the problem

To simplify the problem we shall suppose initially that the fluid is of infinite depth; the modifications required for finite constant depth will be considered in §6 below. It is supposed that the fluid, which is bounded above by a free surface and bounded laterally by parallel vertical sidewalls, is bounded internally by a fixed submerged horizontal cylinder extending from one sidewall to the other. Rectangular Cartesian coordinates are taken so that the horizontal mean free surface is the plane $y = 0$; the y-axis is taken so that the y-coordinate increases with depth. The z-axis is taken parallel to the horizontal generators of the cylinder (the sidewalls are $z = 0$ and $z = \pi/k$), and the x-axis is taken perpendicular to the y-axis and the z-axis. The curve of intersection of the submerged cylinder with any plane $z =$ const. is denoted by C. We shall consider small irrotational fluid motions which are periodic in time; thus we consider velocity potentials of the form $\varphi(x,y)\cos kz\, e^{i\omega t}$, where $\varphi(x,y)$ satisfies the modified Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)\varphi(x,y) = 0 \tag{2.1}$$

in the part of the half-plane $y > 0$ lying outside the curve C. (Solutions of (2.1) will be described as Helmholtz potentials.) On the free surface the condition of constant pressure takes the form

$$K\varphi+\frac{\partial\varphi}{\partial y} = 0 \quad \text{when } y = 0, \tag{2.2}$$

where $K = \omega^2/g$; on the fixed curve C the normal velocity vanishes, i.e.

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on } C. \tag{2.3}$$

We shall be concerned with Helmholtz potentials describing trapping modes (also known as bound states) which are modes of finite total energy and which thus satisfy

$$\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\to 0 \quad \text{when } x^2+y^2\to\infty.$$

It is evident that trapping modes can exist only for certain characteristic frequencies ω, i.e. certain characteristic values of K, which will depend on the curve C and the wavenumber k. As was stated in §1, trapping modes were discovered by Stokes (1846) who showed that on a sloping beach of angle α there is a trapping mode given by

$$\varphi(x,y) = \exp(-kx\cos\alpha-ky\sin\alpha),$$

where the angular frequency ω is given by

$$K = \frac{\omega^2}{g} = k\sin\alpha.$$

Ursell (1952) showed that the Stokes mode on a sloping beach is only the first member of a family: for small angles α there are several modes, the number of modes is finite and increases when α decreases. In the present work we shall be concerned only with mathematical aspects and we shall need to refer to only a few of the many other papers on trapping modes and edge waves that have been published since 1953.

It is instructive now to recall the distinct methods used by Ursell (1951) and Jones (1953) to establish the existence of a trapping mode. We note that the equations and boundary conditions involve two parameters K and k; we investigate trapping modes for which K and k are real, and $0 < K < k$.

(i) Ursell (1951), using Helmholtz multipoles satisfying the free-surface condition (2.2), reduced the problem of a submerged circle to the solution of a homogeneous Fredholm system of equations of the second kind, of the form

$$\xi_m + \sum_1^\infty A_{mn}(K,k)\,\xi_n = 0, \quad m = 1, 2, \ldots,$$

where

$$\sum_{n=1}^\infty \sum_{m=1}^\infty |A_{mn}(K,k)| < \infty.$$

This system can be solved by infinite determinants and has a non-trivial solution if and only if its infinite determinant vanishes, i.e. if

$$\det(\delta_{mn} + A_{mn}(K,k)) = 0, \tag{2.4}$$

in an obvious notation. When k is kept fixed it was found that this determinant has a real root K near k when the radius of the circle is sufficiently small. The coefficients $A_{mn}(K,k)$ are however very complicated, and nothing is known analytically about other roots. (The problem has recently been studied numerically by McIver & Evans 1985). Alternatively the potential could have been represented by a distribution of Helmholtz sources satisfying the free-surface condition (cf. the Appendices at the end of this paper). This represents a trapping-mode potential if the source strength satisfies a homogeneous Fredholm equation of the second kind, i.e. if the Fredholm determinant vanishes. This alternative treatment is applicable to boundary curves C other than circles; for circles it is equivalent to the treatment by multipoles and infinite determinants.

(ii) Jones (1953) treated the parameter K as fixed and the parameter k^2 as an eigenvalue to be determined. The problem then becomes an eigenvalue problem in an appropriate Hilbert space for the Helmholtz equation of acoustics. (In fact, most of Jones's paper is concerned with problems of acoustics.) The corresponding operator is semibounded and self-adjoint; its spectrum is real, consisting of a continuous spectrum with values of k^2 extending from K^2 to $-\infty$, and of a discrete spectrum above K^2. Jones then shows that the discrete spectrum contains at least one point for every submerged boundary curve C symmetrical about $x = 0$. When there is more than one eigenvalue the corresponding trapping modes have different values of k^2 and therefore different lateral wavenumbers and thus refer to different geometrical configurations. In his work Jones uses the difficult spectral theory of unbounded operators applied to partial differential equations in unbounded domains, together with variational methods which are applicable when the eigenparameter appears in the differential equation but not in the boundary conditions.

In the present paper we shall not use such deep results. The parameter k^2 will be kept fixed and the parameter K will be the eigenparameter. It will be sufficient to use the spectral theory of bounded symmetric linear operators together with Kelvin's minimum-energy theorem of classical hydrodynamics.

3. Applications of Kelvin's minimum-energy theorem

In a given domain occupied by an incompressible fluid consider all those motions that have a prescribed normal velocity on the boundary. Kelvin's theorem (Lamb 1932, §45) states that of all these motions the irrotational motion has the smallest kinetic energy. More precisely, the difference between the energy of the irrotational motion and the energy of any other motion is equal to the energy of the difference motion. Several applications of this theorem will now be given, which will be of use to us later, in the proofs of Theorems 4.2, 5.1 and 5.3.

(i) Consider two Helmholtz potentials $\varphi^{(1)}(x,y)$ and $\varphi^{(2)}(x,y)$ which satisfy the same boundary condition

$$\frac{\partial\varphi^{(j)}}{\partial y} = v(x), \quad j = 1,2 \tag{3.1}$$

on the mean free surface $y = 0$. For $j = 1,2$ suppose that the potential $\varphi^{(j)}$ satisfies the boundary condition

$$\frac{\partial\varphi^{(j)}}{\partial n} = 0 \quad \text{on the internal boundary } C^{(j)}. \tag{3.2}$$

Let the corresponding fluid domains be denoted by $D^{(1)}$ and $D^{(2)}$. Suppose now that $D^{(1)} \supset D^{(2)}$ so that $C^{(2)}$ encloses $C^{(1)}$, and consider line integrals of the form

$$\begin{aligned} E(\varphi, D) &= -\int_{\partial D} \varphi\frac{\partial\varphi}{\partial n}\mathrm{d}s \quad \text{taken over the boundary of } D \\ &= \iint_D \{(\nabla\varphi)^2 + k^2\varphi^2\}\,\mathrm{d}x\,\mathrm{d}y > 0. \end{aligned}$$

We note that $E(\varphi, D)$ is proportional to the kinetic energy. In particular,

$$E(\varphi^{(j)}, D^{(j)}) = -\int_{-\infty}^{\infty} \varphi^{(j)}\frac{\partial\varphi^{(j)}}{\partial y}\mathrm{d}x, \quad j = 1,2,$$

where the integral is taken along $y = 0$.

We also note that we can write $E(\varphi^{(2)}, D^{(2)}) = E(\varphi_*^{(2)}, D^{(1)})$, where

$$\begin{aligned} \varphi_*^{(2)} &= \varphi^{(2)} \quad \text{in } D^{(2)}, \\ \varphi_*^{(2)} &= 0 \quad \text{in } D^{(1)} - D^{(2)}; \end{aligned}$$

thus $\varphi_*^{(2)}$ is now defined in the same fluid domain $D^{(1)}$ as $\varphi^{(1)}$ and has the same normal velocity on the boundary of $D^{(1)}$. However, although $\varphi_*^{(2)}$ is irrotational in $D^{(2)}$ and in $D^{(1)} - D^{(2)}$, the tangential velocity is discontinuous along the curve $C^{(2)}$ inside $D^{(1)}$. If Kelvin's theorem can be applied to $\varphi^{(1)}$ and to the discontinuous motion represented by $\varphi_*^{(2)}$ we can conclude that $E(\varphi^{(2)}, D^{(2)}) > E(\varphi^{(1)}, D^{(1)})$. This will now be verified by a direct calculation.

THEOREM 3.1. $E(\varphi^{(2)}, D^{(2)}) > E(\varphi^{(1)}, D^{(1)})$.

Proof. Consider the expression

$$E_{21} = E(\varphi^{(2)}, D^{(2)}) - E(\varphi^{(1)}, D^{(1)}) - E(\varphi^{(2)} - \varphi^{(1)}, D^{(2)}) - E(\varphi^{(1)}, D^{(21)}),$$

where $D^{(21)} = D^{(1)} - D^{(2)}$ is the annular domain bounded by $C^{(1)}$ and $C^{(2)}$, and where the last two terms represent the energy of the difference motion. Then

$$E_{21} = -\int_{-\infty}^{\infty} \varphi^{(2)} \frac{\partial \varphi^{(2)}}{\partial y} \mathrm{d}x + \int_{-\infty}^{\infty} \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial y} \mathrm{d}x + \int_{-\infty}^{\infty} (\varphi^{(2)} - \varphi^{(1)}) \frac{\partial(\varphi^{(2)} - \varphi^{(1)})}{\partial y} \mathrm{d}x$$
$$+ \int_{C^{(2)}} (\varphi^{(2)} - \varphi^{(1)}) \frac{\partial}{\partial n_2} (\varphi^{(2)} - \varphi^{(1)}) \mathrm{d}s + \int_{C^{(2)}} \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial n_{21}} \mathrm{d}s + \int_{C^{(1)}} \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial n_{21}} \mathrm{d}s,$$

where e.g. $\partial\varphi^{(1)}/\partial n_{21}$ denotes the normal gradient into the domain $D^{(21)}$. On using the boundary conditions (3.1) and (3.2) we obtain

$$E_{21} - \int_{-\infty}^{\infty} (\varphi^{(1)} - \varphi^{(2)})\, v(x)\, \mathrm{d}x = -\int_{C^{(2)}} (\varphi^{(2)} - \varphi^{(1)}) \frac{\partial \varphi^{(1)}}{\partial n_2} \mathrm{d}s + \int_{C^{(2)}} \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial n_{21}} \mathrm{d}s$$
$$= -\int_{C^{(2)}} \varphi^{(2)} \frac{\partial \varphi^{(1)}}{\partial n_2} \mathrm{d}s$$
$$= -\int_{C^{(2)}} \varphi^{(1)} \frac{\partial \varphi^{(2)}}{\partial n_2} \mathrm{d}s - \int_{-\infty}^{\infty} \left(\varphi^{(1)} \frac{\partial \varphi^{(2)}}{\partial y} - \varphi^{(2)} \frac{\partial \varphi^{(1)}}{\partial y} \right) \mathrm{d}x,$$

from Green's theorem

$$= \int_{-\infty}^{\infty} (\varphi^{(2)} - \varphi^{(1)})\, v(x)\, \mathrm{d}x,$$

whence $E_{21} = 0$. Thus

$$E(\varphi^{(2)}, D^{(2)}) = E(\varphi^{(1)}, D^{(1)}) + E(\varphi^{(2)} - \varphi^{(1)}, D^{(2)}) + E(\varphi^{(1)}, D^{(21)})$$
$$> E(\varphi^{(1)}, D^{(1)}).$$

This concludes the proof of Theorem 3.1.

(ii) Suppose now that the fluid domain D is bounded internally by the curve C which lies between the vertical lines $x = a$ and $x = b$ where $a < b$. Suppose that $v(x)$ is a function such that

$$\int_{-\infty}^{\infty} |v(x)|^2\, \mathrm{d}x < \infty.$$

In the domain D define the Helmholtz potential $\varphi(x, y)$ satisfying $\partial\varphi/\partial n = 0$ on C, and also $\partial\varphi/\partial y = v(x)$ when $-\infty < x < \infty$ and $y = 0$. We now introduce additional constraints along $x = a$ and $x = b$. Let $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ denote the parts of D in which $-\infty < x < a$, $a < x < b$, $b < x < \infty$ respectively. In the domain $D^{(1)}$ define the potential $\varphi^{(1)}(x, y)$ satisfying $\partial\varphi^{(1)}/\partial x = 0$ when $x = a$, and also $\partial\varphi^{(1)}/\partial y = v(x)$ when $-\infty < x < a$ and $y = 0$; in the domain $D^{(2)}$ define the potential $\varphi^{(2)}(x, y)$ satisfying $\partial\varphi^{(2)}/\partial n = 0$ on C, $\partial\varphi^{(2)}/\partial x = 0$ when $x = a$ and $x = b$, and also $\partial\varphi^{(2)}/\partial y = v(x)$ when $a < x < b$ and $y = 0$; in the domain $D^{(3)}$ define the potential $\varphi^{(3)}(x, y)$ satisfying $\partial\varphi^{(3)}/\partial x = 0$ when $x = b$, and also $\partial\varphi^{(3)}/\partial y = v(x)$ when $b < x < \infty$ and $y = 0$. Then we have

THEOREM 3.2. $E(\varphi, D) < E(\varphi^{(1)}, D^{(1)}) + E(\varphi^{(2)}, D^{(2)}) + E(\varphi^{(3)}, D^{(3)})$.

Proof. Consider the expression

$$E_{123} = E(\varphi^{(1)}, D^{(1)}) + E(\varphi^{(2)}, D^{(2)}) + E(\varphi^{(3)}, D^{(3)}) - E(\varphi, D)$$
$$- E(\varphi^{(1)} - \varphi, D^{(1)}) - E(\varphi^{(2)} - \varphi, D^{(2)}) - E(\varphi^{(3)} - \varphi, D^{(3)}).$$

By an argument similar to the proof of Theorem 3.1 it can then be shown that $E_{123} = 0$. The details are omitted.

(iii) Let $\varphi(x, y)$ denote the same Helmholtz potential as in the last section. Suppose now that the curve C lies below the line $y = l > 0$, and denote by $D^{(4)}$ the strip $-\infty < x < \infty$, $0 < y < l$. In the domain $D^{(4)}$ define the potential $\varphi^{(4)}(x, y)$ satisfying $\partial\varphi^{(4)}/\partial y = 0$ when $-\infty < x < \infty$ and $y = l$, and $\partial\varphi^{(4)}/\partial y = v(x)$ when $-\infty < x < \infty$ and $y = 0$.

Then we have

THEOREM 3.3. $E(\varphi, D) < E(\varphi^{(4)}, D^{(4)})$.

Proof. Consider the expression

$$E_4 = E(\varphi^{(4)}, D^{(4)}) - E(\varphi, D) - E(\varphi^{(4)} - \varphi, D^{(4)}) - E(\varphi, D - D^{(4)}).$$

By an argument similar to the proof of Theorem 3.1 it can then be shown that $E_4 = 0$. The details are omitted.

4. The integral operator and its spectrum

We shall look for trapping modes such that $K < k$; the potential φ is then known to be exponentially small at ∞ (Ursell 1968). In the customary notation we shall write $f(x) \in L_2(-\infty, \infty)$ to indicate that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

and introduce the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$.

Consider the potential

$$\Phi(x, y; v) = -\frac{1}{\pi} \int_{-\infty}^{\infty} v(\xi) \, g(x, y; \xi, 0; C) \, d\xi, \tag{4.1}$$

where $\varphi \in L_2$ and where g is the Green function defined in Appendix A. Then evidently Φ is a Helmholtz potential satisfying $\partial\Phi/\partial n = 0$ on C, and it is easy to see from the definition of the Green function g that

$$\frac{\partial \Phi}{\partial y} = v(x) \quad \text{when } y = 0.$$

The potential $\Phi(x, y; v_0)$ will be a trapping-mode potential if for a certain value of K we have

$$K\Phi + \frac{\partial \Phi}{\partial y} = 0 \quad \text{when } y = 0,$$

i.e.

$$v_0(x) = \frac{K}{\pi} \int_{-\infty}^{\infty} v_0(\xi) \, g(x, 0; \xi, 0; C) \, d\xi \quad \text{when } -\infty < x < \infty. \tag{4.2}$$

Let us write $\lambda = K^{-1}$ and define the operator T_C by the equation

$$T_C \, v = \frac{1}{\pi} \int_{-\infty}^{\infty} v(\xi) \, g(x, 0; \xi, 0; C) \, d\xi, \tag{4.3}$$

then the integral equation (4.2) takes the form

$$(\mathrm{T}_C - \lambda I)\, v_0 = 0;$$

we wish to prove the existence of discrete eigenvalues $\lambda_1, \lambda_2, \ldots$ and eigenfunctions $v_1(x), v_2(x), \ldots \in L_2$.

The familiar Fredholm theory is not applicable to the integral equation which can nevertheless be fully treated because, as we shall see, the operator T_C is symmetric, positive and bounded, i.e. we have

$$g(x,0;\xi,0;C) = g(\xi,0;x,0;C) \tag{4.4}$$

and

$$0 < (\mathrm{T}_C\, v, v) < M_C(v,v) \tag{4.5}$$

for some constant M_C and all $v \in L_2$. By definition the spectrum of the operator T_C consists of those values of λ for which the equation

$$(\mathrm{T}_C - \lambda I)\, u = w \tag{4.6}$$

does not have a unique solution $u \in L_2$ for arbitrary $w \in L_2$; discrete points of the spectrum correspond to trapping modes.

THEOREM 4.1. *The real kernel* $g(\xi_1, 0; \xi_2, 0; C)$ *is symmetric, i.e. the operator* T_C *is symmetric.*

Proof. Apply Green's theorem

$$\int \left(\varphi_1 \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1}{\partial n}\right) \mathrm{d}s = 0$$

to the Helmholtz potentials

$$\varphi_1(x,y) = g(x,y;\xi_1,0;C)$$

and

$$\varphi_2(x,y) = g(x,y;\xi_2,0;C)$$

over the boundary of the domain D indented at $(\xi_1, 0)$ and $(\xi_2, 0)$. The integrand vanishes except at the indentations where it gives

$$g(\xi_1,0;\xi_2,0;C) = g(\xi_2,0;\xi_1,0;C)$$

Thus

$$(\mathrm{T}_C\, v, v) = (v, \mathrm{T}_C\, v).$$

THEOREM 4.2. *The operator* T_C *is bounded, i.e.* $(\mathrm{T}_C\, v, v) < M_C(v,v)$ *for some constant* M_C.

Proof. The scalar product $(\mathrm{T}_C\, v, v)$ is the quantity denoted by $E(\varphi, D)$ in §3 above. By Theorem 3.3 it is sufficient to prove that $(\mathrm{T}_l\, v, v) = E(\varphi^{(4)}, D^{(4)}) < M_l(v,v)$ for some constant M_l, where $\varphi^{(4)}(x,y)$ is the Helmholtz potential which is defined in the strip $-\infty < x < \infty$, $0 < y < l$, and which satisfies the boundary conditions

$$\frac{\partial \varphi^{(4)}}{\partial y} = v(x) \quad \text{when } y = 0, \tag{4.7}$$

$$\frac{\partial \varphi^{(4)}}{\partial y} = 0 \quad \text{when } y = l. \tag{4.8}$$

This can be solved by Fourier transforms, as follows. Write

$$\Phi(\nu, y) = \int_{-\infty}^{\infty} \varphi^{(4)}(x,y)\, \mathrm{e}^{\mathrm{i}\nu x}\, \mathrm{d}x,$$

then

$$\frac{\partial^2 \Phi}{\partial y^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \varphi^{(4)}}{\partial y^2} e^{i\nu x} \, dx = \int_{-\infty}^{\infty} \left(k^2 \varphi^{(4)} - \frac{\partial^2 \varphi^{(4)}}{\partial x^2} \right) e^{i\nu x} \, dx = k^2 \Phi + \nu^2 \int_{-\infty}^{\infty} \varphi^{(4)} e^{i\nu x} \, dx,$$

after two integrations by parts,

$$= (k^2 + \nu^2) \Phi,$$

whence $\Phi(\nu, y) = A(\nu) \cosh\{(l-y)(k^2+\nu^2)^{\frac{1}{2}}\}$, from (4.8). From (4.7) we have

$$\partial \Phi(\nu, 0)/\partial y = -A(\nu)(k^2+\nu^2)^{\frac{1}{2}} \sinh\{l(k^2+\nu^2)^{\frac{1}{2}}\} = \int_{-\infty}^{\infty} v(x) e^{i\nu x} \, dx = W(\nu),$$

say, whence

$$\Phi(\nu, y) = -\frac{W(\nu) \cosh\{(l-y)(k^2+\nu^2)^{\frac{1}{2}}\}}{(k^2+\nu^2)^{\frac{1}{2}} \sinh\{l(k^2+\nu^2)^{\frac{1}{2}}\}}$$

and in particular $$\Phi(\nu, 0) = -\frac{W(\nu)}{(k^2+\nu^2)^{\frac{1}{2}}} \coth\{l(k^2+\nu^2)^{\frac{1}{2}}\}.$$

We have $$(T_l v, v) = -\int_{-\infty}^{\infty} \varphi^{(4)}(x, 0) v(x) \, dx$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\nu, 0) W(\nu) \, d\nu,$$

by Parseval's theorem,

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |W(\nu)|^2 \coth\{l(k^2+\nu^2)^{\frac{1}{2}}\} \frac{d\nu}{(k^2+\nu^2)^{\frac{1}{2}}},$$

and similarly $$(v, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |W(\nu)|^2 \, | \, d\nu.$$

Also $$\frac{\coth\{l(k^2+\nu^2)^{\frac{1}{2}}\}}{(k^2+\nu^2)^{\frac{1}{2}}} < \frac{1}{k} \coth kl$$

for all real ν, and it follows at once that

$$(T_C v, v) < (T_l v, v) < \frac{1}{k} \coth kl \, (v, v). \tag{4.9}$$

This completes the proof of Theorem 4.2.

THEOREM 4.3. *The spectrum of* T_C *is real, continuous for* $0 < \lambda < k^{-1}$ *and discrete for* $k^{-1} < \lambda < M_C$.

Proof. We note from the equation

$$(T_C v, v) = -\int_{-\infty}^{\infty} \varphi \frac{\partial \varphi}{\partial y} dx = \iint_D \{(\nabla \varphi)^2 + k^2 \varphi^2\} \, dx \, dy \tag{4.10}$$

that T_C is positive. Since T_C is symmetric and bounded the spectrum is real (see Riesz & Sz-Nagy 1952, § 107). To investigate the spectrum we must investigate the solution of the equation

$$(T_C - \lambda I) u = w, \tag{4.11}$$

when $w \in L_2$ and λ is not real, and then let λ tend to a real value. It is readily seen (cf. (4.1)) that, when $\cos \alpha = (l\lambda)^{-1}$ is not real, the solution is

$$u(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} w(\xi) G(x, 0; \xi, 0; C; k \cos \alpha) \, d\xi, \tag{4.12}$$

where $G(\ )$ is the Green function defined in Appendix B. Suppose first that $\operatorname{Re}(k\lambda)^{-1} < 1$ and let λ tend to a real value. It is shown in Appendix B that the Green function tends to a real limit function except at a discrete set of real values of λ, and that this limit function is exponentially small when $|x| \to \infty$. It is then not difficult to show that $u(x)$ defined by (4.12) belongs to L_2 whenever $G(\ )$ exists; thus the spectrum of T_C is discrete when $\lambda < k^{-1}$. Suppose next that $\operatorname{Re}(k\lambda)^{-1} > 1$, $\operatorname{Im}(k\lambda)^{-1} > 0$, and let λ tend to a real value, then the Green function can be shown to tend to a complex-valued limit function except at a discrete set of values λ, but now this limit function is wavelike at ∞, and $u(x)$ defined by (4.12) does not belong to L_2. Thus the spectrum of T_C is continuous when $0 < \lambda < k^{-1}$.

5. Existence of a trapping mode

Since T_C is a real bounded symmetric operator it has a spectral decomposition (Riesz & Sz-Nagy 1951, §107), and, since the discrete spectrum lies above the continuous spectrum in the finite interval $k^{-1} < \lambda < M_C$, the eigenvalues in this interval can be found by the familiar variational principles. (For a detailed justification see Weinstein & Stenger 1972.) In particular, the largest eigenvalue in this interval is given by

$$\lambda_1 = \max \frac{(\mathrm{T}_C u, u)}{(u, u)} \tag{5.1}$$

taken over all real $u \in L_2$, where the maximum is actually attained by the corresponding eigenfunction. We can now prove the following theorem.

THEOREM 5.1. *Suppose that $C^{(1)}$ lies inside $C^{(2)}$, and that the eigenvalue problem for $C^{(1)}$ has p eigenvalues such that $\lambda_1^{(1)} \geqslant \lambda_2^{(1)} \geqslant \ldots \geqslant \lambda_p^{(1)} > k^{-1}$. Then the eigenvalue problem for $C^{(2)}$ has at least p eigenvalues, and $\lambda_s^{(2)} > \lambda_s^{(1)}$ when $s = 1, 2, \ldots, p$.*

Proof. Denote the corresponding operators by $\mathrm{T}^{(1)}$ and $\mathrm{T}^{(2)}$. Then $\lambda_1^{(j)} = \max(\mathrm{T}^{(j)}u, u)/(u, u)$, $j = 1, 2$; also $(\mathrm{T}^{(2)}u, u) > (\mathrm{T}^{(1)}u, u)$ by Theorem 3.1. It follows that $\lambda_1^{(2)} > \lambda_1^{(1)}$. For let the corresponding eigenfunction for $C^{(1)}$ be denoted by $u_1^{(1)}$. Then

$$\begin{aligned} \lambda_1^{(1)} = \frac{(\mathrm{T}^{(1)}u_1^{(1)}, u_1^{(1)})}{(u_1^{(1)}, u_1^{(1)})} &< \frac{(\mathrm{T}^{(2)}u_1^{(1)}, u_1^{(1)})}{(u_1^{(1)}, u_1^{(1)})} \\ &\leqslant \max \frac{(\mathrm{T}^{(2)}u, u)}{(u, u)} = \lambda_1^{(2)}. \end{aligned} \tag{5.2}$$

The corresponding inequalities for the higher eigenvalues follow from the familiar minimax generalization of this argument (Courant & Hilbert 1931, chapter 6). The details are omitted.

THEOREM 5.2. *Suppose that C is any submerged contour enclosing a finite area. Then there exists at least one trapping mode.*

Proof. Choose a point P inside C, and consider circles $(C^{(1)}(R))$ with centre P and radius R. It is known (Ursell 1951) that there is at least one trapping mode for such circles provided that R is sufficiently small; $R < R_0(P)$ say. Now choose $R_1 < R_0(P)$ so that $C^{(1)}(R_1)$ lies inside C. The existence of a trapping mode now follows from Theorem 5.1. An alternative proof can be based on Jones (1953) where the existence of a trapping mode is shown for a wide class of curves C including circles. In Jones's work it is $K = (\lambda)^{-1} > 0$ that is prescribed and the existence of an eigenvalue k that is deduced; $k = \kappa_0(K)$, say. Also $\kappa_0(K) > K$ for all K, and κ_0 is close to K when the radius of the circle is small. We can now deduce that there is a trapping mode when

k is prescribed; $k = k_0$, say. Choose $K' = (\lambda')^{-1}$ so that $K' < k_0$, and take the radius of the circle so small that $\kappa_0(K') < k_0$. Let K increase from K' to k_0; since $\kappa_0(K) > K$ it follows by continuity that $\kappa_0(K)$ takes the value k_0 for some value of K between K' and k_0.

THEOREM 5.3. *For any given submerged curve C there are at most finitely many trapping modes.*

Proof. Let the fluid domain D be dissected into domains $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ by vertical lines $x = a$ and $x = b$, as in Theorem 3.2. In the domain $D^{(2)}$ of finite horizontal extent all the eigenvalues must be discrete; let them be denoted by $\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots$, (where $\lambda_m^{(2)} \to 0$ when $m \to \infty$), with corresponding eigenfunctions $v_1^{(2)}, v_2^{(2)}, \ldots$. Then it is easy to show that

$$\int_a^b v_p^{(2)} v_q^{(2)}\,\mathrm{d}x = 0 \quad \text{when } \lambda_p^{(2)} \neq \lambda_q^{(2)}.$$

We shall normalize so that

$$\int_a^b v_p^{(2)} v_q^{(2)}\,\mathrm{d}x = \delta_{pq} = \begin{cases} 0, & p \neq q, \\ 1, & p = q, \end{cases}$$

for all p and q. For any $v^{(2)}(x)$ (with $a < x < b$) write

$$v^{(2)}(x) = \sum_1^s \alpha_m v_m^{(2)} + V_{s+1}^{(2)}, \quad \text{where } \alpha_m = (v^{(2)}, v_m^{(2)})$$

and where $V_{s+1}^{(2)}$ is orthogonal to $v_m^{(2)}$, $m = 1, 2, \ldots, s$. Let the operator analogous to T for the domain $D^{(2)}$ be denoted by $\mathrm{T}^{(2)}$. Then

$$\mathrm{T}^{(2)} v_m^{(2)} = -\varphi_m^{(2)} = \lambda_m^{(2)} v_m^{(2)}.$$

Thus
$$\mathrm{T}^{(2)} v^{(2)} = \sum_1^s \alpha_m \lambda_m^{(2)} v_m^{(2)} + \mathrm{T}^{(2)} V_{s+1}^{(2)}$$

and
$$(\mathrm{T}^{(2)} v^{(2)}, v^{(2)}) = \sum_1^s \lambda_m^{(2)} \alpha_m^2 + (\mathrm{T}^{(2)} V_{s+1}^{(2)}, V_{s+1}^{(2)})$$

since $(V_{s+1}^{(2)}, v_m^{(2)}) = 0$ and

$$(\mathrm{T}^{(2)} V_{s+1}^{(2)}, v_m^{(2)}) = (V_{s+1}^{(2)}, \mathrm{T}^{(2)} v_m^{(2)}) = \lambda_m^{(2)} (V_{s+1}^{(2)}, v_m^{(2)}) = 0.$$

Also
$$(\mathrm{T}^{(2)} V_{s+1}^{(2)}, V_{s+1}^{(2)}) \leqslant \lambda_{s+1}^{(2)} (V_{s+1}^{(2)}, V_{s+1}^{(2)}) = \lambda_{s+1}^{(2)} (v^{(2)}, v^{(2)}) - \lambda_{s+1}^{(2)} \sum_1^s \alpha_m^2$$

and so
$$(\mathrm{T}^{(2)} v^{(2)}, v^{(2)}) \leqslant \lambda_{s+1}^{(2)} (v^{(2)}, v^{(2)}) + \sum_1^s (\lambda_m^{(2)} - \lambda_{s+1}^{(2)}) \alpha_m^2. \tag{5.3}$$

Consider next the domain $D^{(3)}$ and the potential $\varphi^{(3)}$, see Theorem 3.2. Since $\partial\varphi^{(3)}/\partial x = 0$ when $x = b$ the boundary-value problem for $\varphi^{(3)}$ can be solved by Fourier cosine transforms. The calculation is similar to the calculation in Theorem 4.1; the details are omitted. We thus find that

$$(\mathrm{T}^{(3)} v^{(3)}, v^{(3)}) < k^{-1} (v^{(3)}, v^{(3)}) \tag{5.4}$$

and similarly that

$$(\mathrm{T}^{(1)} v^{(1)}, v^{(1)}) < k^{-1} (v^{(1)}, v^{(1)}).$$

According to Theorem 3.2,

$$(\mathrm{T}v, v) < (\mathrm{T}^{(1)}v^{(1)}, v^{(1)}) + (\mathrm{T}^{(2)}v^{(2)}, v^{(2)}) + (\mathrm{T}^{(3)}v^{(3)}, v^{(3)}),$$

and therefore

$$(\mathrm{T}v, v) < k^{-1}(v^{(1)}, v^{(1)}) + k^{-1}(v^{(3)}, v^{(3)}) + \lambda_{s+1}^{(2)}(v^{(2)}, v^{(2)}) + \sum_{m=1}^{s} (\lambda_m^{(2)} - \lambda_{s+1}^{(2)})(v^{(2)}, v_m^{(2)})^2. \tag{5.5}$$

Theorem 5.3 now follows from the following theorem.

THEOREM 5.4. *Suppose that s is chosen so that $\lambda_{s+1}^{(2)} \leqslant k^{-1} < \lambda_s^{(2)}$. Then there are at most s trapping modes.*

Proof. Suppose that there are $(s+1)$ trapping modes $v_1, v_2, \ldots, v_{s+1}$; choose the combination $y = \Sigma_1^{s+1} \beta_q v_q$ so that $(y, v_m^{(2)}) = 0$ when $m = 1, 2, \ldots, s$. This is possible because there are s equations in $(s+1)$ unknowns. From (5.5) we now have

$$(\mathrm{T}y, y) < k^{-1}(y^{(1)}, y^{(1)}) + \lambda_{s+1}^{(2)}(y^{(2)}, y^{(2)}) + k^{-1}(y^{(3)}, y^{(3)}) < k^{-1}(y, y). \tag{5.6}$$

On the other hand we have

$$(\mathrm{T}y, y) = (\Sigma \beta_p \mathrm{T}v_p, \Sigma \beta_q v_q) = (\Sigma \lambda_p \beta_p v_p, \Sigma \beta_q v_q).$$

It is however easy to see that $(v_p, v_q) = 0$ when $\lambda_p \neq \lambda_q$; we choose $(v_p, v_q) = \delta_{pq}$. Then $(\mathrm{T}y, y) = \Sigma_1^{s+1} \lambda_p \beta_p^2$ and similarly $(y, y) = \Sigma_1^{s+1} \beta_p^2$, and therefore

$$(\mathrm{T}y, y) > k^{-1}(y, y), \tag{5.7}$$

since $\lambda_p > k^{-1}$ for all trapping modes. The inequalities (5.6) and (5.7) are contradictory, and Theorem 5.4 follows. This concludes the proof of theorem 5.3 which is similar to the proof of Theorem 1 of Jones (1953).

Theorem 5.2 has shown that there is at least one trapping mode, while Theorem 5.4 has given an upper bound for the number of trapping modes. If the existence of two or more trapping modes could be proved for a certain curve C it would then be possible to prove the existence of at least the same number of modes for all curves enclosing C, by Theorem 5.1. At present the only example for which the number of trapping modes is exactly known is the sloping beach (Ursell 1952) which is unsuitable because the boundary is of infinite length (see however §7 below).

6. Extension to finite depth

So far it has been assumed that the depth of the fluid is infinite but the same arguments remain applicable to finite constant depth h, provided that it can be shown that in fluid of finite constant depth there is at least one trapping mode for an arbitrarily small submerged circle. This, however, follows from Jones (1953) together with a continuity argument. (The corresponding argument for infinite depth is given as an alternative argument in the proof of Theorem 5.2.) An alternative proof for the small circle can probably be constructed by the method of Ursell (1951). Similar considerations apply to humps on the bottom for which the existence of a trapping mode was proved by Jones (1953).

7. Discussion

The solution for the sloping beach suggests that for curves close to the free surface we may expect a large number of trapping modes. The following argument gives some support to this suggestion. To fix ideas let us treat only the case of infinite depth.

We consider a sequence of circles

$$C_m;\ x^2+(y-f_m)^2 = (l-f_m)^2,$$

where $f_1 > l$ and where (f_m) is an increasing sequence tending to ∞. All these circles touch the line $y = l$ from below at the point $(0, l)$ and the circles tend to the line $y = l$ when $f_m \to \infty$. We have seen that for each of these circles the spectrum consists of the line segment $0 < \lambda < k^{-1}$, together with a finite number N_m of discrete points along the segment $k^{-1} < \lambda < k^{-1} \coth kl$ (see Theorem 4.2). Theorem 5.1 shows that $N_{m+1} \geqslant N_m$; we now wish to show that $N_m \to \infty$. Denote by $\mathrm{T}^{(m)}$ the operator corresponding to the circle C_m. It can be shown that the operator $\mathrm{T}^{(m)}$ tends strongly to a limit operator $\mathrm{T}^{(\infty)}$ when $m \to \infty$ (Riesz & Sz-Nagy 1951, §104). Let us additionally assume that $\mathrm{T}^{(\infty)}$ is the operator corresponding to the limit of the circles C_m, i.e. to the line $y = l$, and that the spectrum of $\mathrm{T}^{(m)}$ tends to the spectrum of $\mathrm{T}^{(\infty)}$. We now note that the spectrum of the limit problem consists of all the points of the segment $0 < \lambda < k^{-1} \coth kl$, as is easily seen (cf. the proof of Theorem 4.2), and this cannot be the limit of the spectrum of $\mathrm{T}^{(m)}$ unless $N_m \to \infty$. This argument remains applicable, with obvious minor modifications, to trapping modes in fluid of finite constant depth.

Let us next consider trapping modes when the curve C is symmetrical about the line $x = 0$. Evidently each trapping mode must be either an odd or even function of x. The mode constructed for the small submerged circle by Ursell (1951) is an even mode, and the existence of odd modes for a sufficiently large circle has not yet been rigorously proved although such a mode for one such circle has been found numerically by P. A. Martin (1985, unpublished note). The preceding argument can be used to make the existence of odd modes very plausible. In the proof of Theorem 4.2 let the normal velocity $v(x)$ be an odd function of x, and let us again consider the sequence of circles C_m. The spectrum of the limit problem for odd $v(x)$ again consists of all the points of the segment $0 < \lambda < k^{-1} \coth kl$. If there are no odd trapping modes the segment $k^{-1} < \lambda < k^{-1} \coth kl$ will be empty for all m, and this again leads to a contradiction. In fact this argument shows that the number of odd modes, like the number of even modes, tends to ∞ when $m \to \infty$.

The argument of the present paper may possibly be extended to submerged bodies that are not cylindrical. We have seen that for a cylinder there is at least one trapping mode which is antisymmetrical about the plane $z = \pi/2k$. Let the cylinder be expanded into a submerged non-cylindrical surface which is symmetrical about $z = \pi/2k$, and let us consider modes which are antisymmetrical about $z = \pi/2k$. The modified Helmholtz equation (2.1) must now be replaced by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi(x, y, z) = 0,$$

but the spectrum is still discrete in a certain interval and by Kelvin's theorem it contains at least one point. This argument suggests that there is at least one trapping mode (antisymmetrical about $z = \pi/2k$) for any submerged sphere with its centre on $z = \pi/2k$.

Evans & McIver (1984) have studied trapping modes over a rectangular shelf of finite length lying on the bottom of fluid of finite constant depth. They use Jones's formulation of the problem, i.e. they treat K as a prescribed parameter, and they can thus give both upper and lower bounds for the number of trapping modes. They also solve this problem numerically by solving a singular homogeneous integral equation of the first kind and find that Jones's bounds are very effective: the number of trapping modes estimated by them differs by at most one from the correct number. However, in Jones's problem the prescribed parameter is K and the corresponding values of k are to be found, whereas in our problem the prescribed parameter is k and the corresponding values of K are to be found. Thus a direct comparison of Jones's results and our results is not possible. McIver & Evans (1985) have also made a numerical study of the submerged circle by the method of Ursell (1951); trapping modes exist whenever a certain infinite determinant vanishes. Again it is the parameter K that they treat as prescribed.

Another approach to the problem of trapping modes has recently been put forward by Aranha (1986). Aranha obtains an equation of the form

$$F(\lambda, k)\, u = 0 \tag{7.1}$$

(cf. (2.4)) but not of the form $(\mathrm{T}-\lambda I)\, u = 0$. Thus the spectral theory of self-adjoint operators and the variational principle of §5 above are not applicable to (7.1). I have not been able to follow Aranha's argument which is given only in outline, but his conclusions, which are similar to mine, led me to the present re-examination of the problem of trapping modes.

Appendix A. Construction of the Green function $g(x, y;\, \xi, 0;\, C)$ for infinite depth

The function $g(x, y;\, \xi, 0;\, C)$ satisfies the modified Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right) g = 0 \tag{A 1}$$

in the fluid, with the boundary conditions

$$\frac{\partial g}{\partial y} = 0 \quad \text{when } y = 0 \quad \text{except at } (\xi, 0), \tag{A 2}$$

$$g \to 0 \quad \text{when } x^2+y^2 \to \infty, \tag{A 3}$$

and

$$\frac{\partial g}{\partial n} = 0 \quad \text{on } C; \tag{A 4}$$

the function g has a source singularity at $(\xi,\, 0)$, i.e.

$$g(x, 0;\, \xi, 0;\, C) - K_0\{k((x-\xi)^2+y^2)^{\frac{1}{2}}\} \quad \text{is bounded near } (x, y) = (\xi, 0). \tag{A 5}$$

Here

$$K_0(Z) = \int_0^\infty \exp\,(-Z \cosh \mu)\, \mathrm{d}\mu$$

is the usual Bessel function, such that $K_0(Z) \sim (\pi/2Z)^{\frac{1}{2}}\, \mathrm{e}^{-Z}$ when $Z \to \infty$ and $K_0(Z) \sim -\log Z$ when $Z \to 0$. To construct this Green function we consider

$$\begin{aligned} g_*(x, y;\, \xi, 0;\, C) &= K_0\{k((x-\xi)^2+y^2)^{\frac{1}{2}}\} \\ &\quad + \oint m(s)\,[K_0\{k((x-X(s))^2+(y-Y(s))^2)^{\frac{1}{2}}\} + K_0\{((x-X(s))^2+(y+Y(s))^2)^{\frac{1}{2}}\}]\,\mathrm{d}s, \end{aligned} \tag{A 6}$$

where s is the arc length along C, where the points of C have coordinates $X(s)$, $Y(s)$, and where $m(s)$ is to be determined. Evidently the function g_* satisfies all the conditions except the condition on C, and this is also satisfied if $m(s)$ can be chosen so that, when $(x,y) = (X(s'), Y(s'))$, we have

$$-\frac{\partial}{\partial n}K_0\{(k(x-\xi)^2+y^2)^{\frac{1}{2}}\} = -\pi m(s') + \oint m(s)\frac{\partial}{\partial n}\left[\begin{array}{l} K_0\{k((x-X(s))^2+(y-Y(s))^2)^{\frac{1}{2}}\} \\ +K_0\{k((x-X(s))^2+(y+Y(s))^2)^{\frac{1}{2}}\}\end{array}\right]\mathrm{d}s. \quad \text{(A 7)}$$

This is a Fredholm integral equation of the second kind. To prove that it always has a unique solution we need to show that the corresponding homogeneous equation has no solution except $m(s) \equiv 0$. The proof of this result is similar to the proof of the corresponding result for the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\kappa^2\right)\Phi(x,y) = 0,$$

see e.g. Ursell (1973, §2). The homogeneous equation has a non-trivial solution only when κ^2 is an eigenvalue of the interior Dirichlet problem. Since these eigenvalues are known to be positive and we are concerned with the modified Helmholtz equation (A 1) it follows that (A 7) can always be uniquely solved. This concludes the proof of the existence of the Green function for infinite depth.

For finite depth the function

$$K_0\{((x-\xi)^2+y^2)^{\frac{1}{2}}\} = \int_0^\infty \exp(-ky\cosh\mu)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu \quad \text{(A 8)}$$

is replaced by an expression of the form

$$g_1(x,y;\xi,0;C;h) = K_0\{k((x-\xi)^2+y^2)^{\frac{1}{2}}\} + \int_0^\infty A(\mu)\cosh(ky\cosh\mu)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu \quad \text{(A 9)}$$

where $A(\mu)$ is chosen so that $\partial g_1/\partial y = 0$ when $y = h$, whence

$$-k\sinh\mu\exp(-kh\cosh\mu)+k\sinh\mu A(\mu)\sinh(kh\cosh\mu) = 0,$$

i.e.

$$A(\mu) = \frac{\exp(-kh\cosh\mu)}{\sinh(kh\cosh\mu)}.$$

Evidently $\partial g_1/\partial y = 0$ when $y = 0$. The sum of the two Bessel functions in the kernel is similarly modified and the construction of the corresponding Green functions for finite depth can be completed as before.

Appendix B. Construction of the Green function $G(x,y;\xi,0;C;K)$ for infinite depth

This Green function satisfies the same equations (A 1), (A 3), (A 4) as the Green function $g(x,y;\xi,0;C)$, but (A 2) is replaced by the boundary condition

$$KG+\frac{\partial G}{\partial y} = 0 \quad \text{when } y = 0. \quad \text{(B 1)}$$

It is assumed that K is not real, and that $\operatorname{Re} K < k$ and we shall write $K = k\cos\alpha$, where $\mathrm{Re}\,(\cos\alpha) < 1$. When C is absent the potential of a source singularity at $(\xi, 0)$ subject to (B 1) is given by $G_1(x, y;\ \xi, 0;\ K)$, where

$$\left(K+\frac{\partial}{\partial y}\right)G_1 = \frac{\partial}{\partial y}K_0\{((x-\xi)^2+y^2)^{\frac{1}{2}}\},$$

i.e. by

$$\left(k\cos\alpha+\frac{\partial}{\partial y}\right)G_1 = \frac{\partial}{\partial y}\int_0^\infty \exp\left(-k\cosh\mu\right)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu$$

$$= -k\int_0^\infty \cosh\mu\exp\left(-ky\cosh\mu\right)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu,$$

see Ursell (1968, §2). Thus

$$G_1 = \int_0^\infty \frac{\cosh\mu}{\cosh\mu-\cos\alpha}\exp\left(-ky\cosh\mu\right)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu. \qquad \text{(B 2)}$$

It can be shown that G_1 is exponentially small at ∞. We can also construct the potential $G_2(x, y;\ X, Y;\ K)$ of a submerged source singularity $K_0\{k((x-X)^2+(y-Y)^2)^{\frac{1}{2}}\}$ at $(x, y) = (X, Y)$. We find (see Ursell 1951, equation 10), that

$$G_2(x, y;\ X;\ Y) = K_0\{k((x-X)^2+(y-Y)^2)^{\frac{1}{2}}\}$$
$$+\int_0^\infty \frac{\cosh\mu+\cos\alpha}{\cosh\mu-\cos\alpha}\exp\{-k(y+Y)\cosh\mu\}\cos\{k(x-X)\sinh\mu\}\,\mathrm{d}\mu. \qquad \text{(B 3)}$$

We note that evidently the denominator $\cosh\mu-\cos\alpha = \cosh\mu-(K/k)$ in (B 2) and (B 3) does not vanish. To construct the Green function $G(x, y;\ \xi, 0;\ C;\ K)$ we write

$$G(x, y;\ \xi, 0;\ C, K) = G_1(x, y;\ \xi, 0;\ K)+\oint_C M(s)\,G_2(x, y;\ X(s), Y(s);\ K)\,\mathrm{d}s \qquad \text{(B 4)}$$

where $M(s)$ satisfies the Fredholm equation of the second kind

$$-\frac{\partial}{\partial n}G_1(x, y;\ \xi, 0;\ K) = -\pi M(s')+\oint M(s)\frac{\partial}{\partial n}G_2(x, y;\ X(s), Y(s);\ K)\,\mathrm{d}s \qquad \text{(B 5)}$$

when $(x, y) = (X(s'), Y(s'))$. This equation has a unique solution unless the Fredholm determinant vanishes, in which case there is a function $M_0(s) \not\equiv 0$ such that

$$-\pi M_0(s')+\int M_0(s)\frac{\partial}{\partial n}G_2(x, y;\ X(s), Y(s);\ K)\,\mathrm{d}s = 0 \qquad \text{(B 6)}$$

when $(x, y) = (X(s'), Y(s'))$; but this is impossible unless K is real. For consider the Helmholtz potential

$$\Psi(x, y) = \oint M_0(s)\,G_2(x, y;\ X(s), Y(s), K)\,\mathrm{d}s, \qquad \text{(B 7)}$$

which evidently satisfies the boundary conditions

$$K\Psi+\frac{\partial\Psi}{\partial y} = 0 \quad \text{when } y = 0$$

and
$$\frac{\partial\Psi}{\partial n} = 0 \quad \text{on } C,$$

and which is exponentially small at ∞. Let Ψ^* denote the complex conjugate of Ψ, then by Green Theorem

$$-\int \Psi^* \frac{\partial \Psi}{\partial n}\,\mathrm{d}s = \iint \Psi^* \nabla^2 \Psi \,\mathrm{d}x\,\mathrm{d}y + \iint |\nabla \Psi|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

i.e.

$$K\int_{-\infty}^{\infty} |\Psi|^2\,\mathrm{d}x = k^2 \iint |\Psi|^2\,\mathrm{d}x\,\mathrm{d}y + \iint |\nabla \Psi|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, if K is not real, we must have $\Psi(x,y) \equiv 0$ outside C. Now consider the potential $\Psi(x,y)$ defined by (B 7) for points (x,y) inside C; by an argument similar to the argument quoted in Appendix A we can now show that $\Psi(x,y) \equiv 0$ in C, and it follows that $M_0(s) \equiv 0$, a contradiction. It follows that K must be real if the Fredholm determinant is to vanish. Since $\operatorname{Re} K < k$ the kernel of the integral equation is analytic in K (because $\cosh\mu - \cos\alpha$ does not vanish), therefore the Fredholm determinant is also analytic in K and can vanish for at most a discrete set of values of K. If there are such zeros then the construction for G fails but the function Ψ given by (B 7) is then the potential of a trapping mode. It follows that the spectrum is discrete when $K < k$, i.e. when $k^{-1} < \lambda$. By a similar argument we can show that the spectrum is continuous when $k < K$. For suppose that $\operatorname{Re} K > k$ and $\operatorname{Im} K > 0$, then the integrands in the expressions for the source functions G_1 and G_2 have a pole above the real μ-axis which tends to the real axis when K tends to a real value. Let the contour of integration in the μ-plane be deformed to pass below the real μ-axis near the pole. It is then evident that in the limit when K is real the functions G_1 and G_2 tend to complex-valued limit functions which are analytic functions of K. (When K approaches the limit from below the corresponding limit functions are the complex conjugates.) The construction of G now proceeds as before and can fail for at most a discrete set of real values of K. It is evident, however, that the behaviour of G is wavelike when $|x| \to \infty$, and that therefore the function $u(x)$ defined by (4.12) does not belong to L_2. It follows that the spectrum is continuous when $K > k$, i.e. when $0 < \lambda < k^{-1}$. The construction for finite depth h is similar, thus the potential of a source singularity at $(\xi, 0)$ now has the form

$$G_h = \int_0^{\infty} \frac{\cosh\mu}{\cosh\mu - \cos\alpha} \exp(-ky\cosh\mu)\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu$$
$$+\int_0^{\infty} \{B(\mu)\cosh(ky\cosh\mu) + C(\mu)\sinh(ky\cosh\mu)\}\cos\{k(x-\xi)\sinh\mu\}\,\mathrm{d}\mu, \quad \text{(B 8)}$$

where $B(\mu)$ and $C(\mu)$ are chosen so that

$$\left(k\cos\alpha + \frac{\partial}{\partial y}\right)G_h = 0 \quad \text{when } y = 0$$

and

$$\frac{\partial}{\partial y} G_h = 0 \quad \text{when } y = h;$$

the potential for a submerged source can be constructed in the same way, and the Green function can then be obtained by solving an integral equation corresponding to (B 5) above. The denominators in the source potentials are found to contain a factor

$$\cos\alpha\cosh(kh\cosh\mu) - \cosh\mu\sinh(kh\cosh\mu),$$

and it follows that the spectrum is discrete when $\lambda > (k \tanh kh)^{-1}$ and continuous below this value where we note that

$$(k \tanh kh)^{-1} > k^{-1}.$$

The details of these calculations are lengthy but straightforward and are omitted.

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